# A Generalized 3D Subgrid Technique for the Finite-Difference Time Domain Method

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*Abstract*—**A new subgridding technique for finite difference (FD) methods is presented. The method is based on the integral form of Maxwell's equations combined with a simple yet efficient orthogonalization technique for the discretization geometry at subgrid interfaces. No additional correction factors or interpolations are required. This leads to spurious-mode free solutions when applied to FD approximations of eigenvalue problems and to stable difference formulations when applied to the finite difference time-domain (FD-TD) method. The high efficiency of the subgridding technique is demonstrated by the FD-TD analysis of an inter-digital filter with circular posts.**

## I. Introduction

Due to its high flexibility, the finite difference time domain (FD-TD) method is well established for solving a wide variety of electromagnetic problems [1] - [12]. Based on Yee's cell [1], combined with graded mesh techniques [11], [12], a localized accurate modeling of microwave structures with non-rectangular sections or portions of high field variations leads often to an unnecessarily fine mesh discretization in homogeneous areas of low field gradients, due to the topology of the grid. Moreover, in many cases the stable time step depends on the smallest cell used, and still rather high memory and computation time requirements are usually necessary.

A more universal approach is to use locally refined meshes [2], [3], [6]. The reported subgrid FD-TD algorithms, however, require additional interpolation schemes at the grid-interfaces. Depending on the type of interpolation, this may violate the divergence relations, which often results in unstable formulations.

In this paper, we introduce a simple yet efficient orthogonalization technique for the discretization geometry at the grid-interfaces. A FD-TD formulation for generalized coordinates is applicable which does not require any additional interpolation. Own numerical tests have shown that the stability of this method depends on a Courant-Friedrichs-Lewy (CFL) type stability condition [14] which includes the deformation of a cell. The time step is localized and matched to the local CFL-condition in each grid-level. The discretization of the time differentiation on grid interfaces is performed such that the divergence (i.e. the surface integration along the cell surface over all contributions to the normal field components on the cell faces) vanishes after one time step in the grid-level containing the cell. This leads to a general, efficient and stable subgridding technique. The example of an inter-digital filter where the filter characteristics are highly sensitive to the adequate discretization of the capacitive gap sections verifies the proposed method by good agreement with measurements.

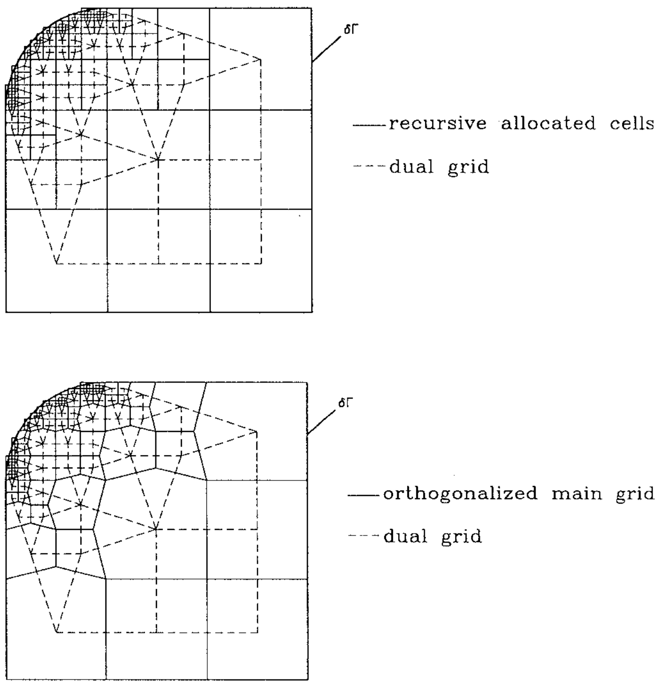


Fig. 1. 2D subgrid generation of a FD-TD domain with the contour $\delta\Gamma$. (a) Recursive allocated cells (solid lines), dual grid (dashed lines). (b) Orthogonalized main grid (solid lines), dual grid (dashed lines)

## II. Theory

**Grid generation**–Based on a 2 : 1 cell ratio for the subgrids[1], a recursive grid-generation procedure is utilized (cf. Figs. 1). A given object (for instance a domain bounded by a contour $\delta\Gamma$ cf. Fig. 1a) is first subdivided

[1] Further refinement is straightforward by inserting subgrids into subgrids

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into cells of a largest level $l$ such that all corners of the cell are still inside the object. In order to discretize the remaining space, bounded by the object and the level $l$ cells, the same procedure is repeated using cells of the next finer level $l+1$, until the maximum specified level $l_m$ is obtained.

After this cell allocation procedure has been performed for the specified objects, a *subgrid boundary-condition* is applied. This condition forces each cell of level $l$ to have neighbor cells in the levels $l-1$, $l$ and $l+1$ only. From the allocated cells, two different grids are derived, which are referred to as the main and the dual grid. The main grid is defined by the corners, and the dual grid is defined by the centers of the allocated cells (Fig. 1a). The dual mesh, as defined above, involves quadrangular and triangular cells.

A direct discretization of the integral form of Maxwell's equations for non-orthogonal grids (for example according to [8]) applied to this scheme usually requires interpolations of components which are not defined in the subgrid geometry. Therefore, a new technique is introduced where the main grid is adequately orthogonalized against the dual grid (Fig. 1b). The result of this orthogonalization procedure is shown in Fig. 1(b). In the three dimensional (3D) case, the orthogonalization of the main and the dual grid is performed analogously, as shown in Fig. 2, which depicts a general subgrid interface. Here, a nonlinear equation has to be solved for the coordinates of the orthogonalized main grid.

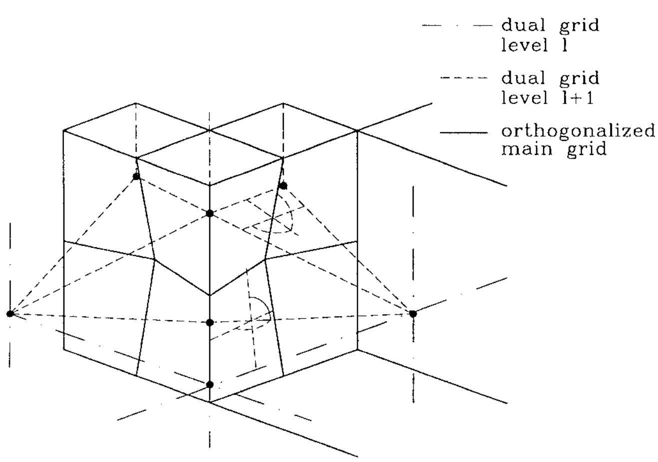



Fig. 2. 3D subgrid generation of a FD-TD domain. Dual grid level $l$ (dash-dotted lines), dual grid level $l+1$ (dashed lines), orthogonalized main grid (solid lines)

**3D Finite-Difference-Equations** – The main grid of the 3D subgrid discretization defines local basis-systems $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$. The electric and the magnetic fields are represented by contra-variant components ${}^l e^i$, ${}^l h^i$, $i = 1 \dots 3$ as unknowns, where the superscript $^l$ designates the grid level, $0 < l < l_m$ of the cell, related to the component, and $l_m$ is the maximal (finest) grid level. The time increment in the grid of level $l$ is $2^{l_m - l}\Delta t$, where $\Delta t$ is the time increment in level $l_m$. The electric and the magnetic field-components in level $l$ are defined at the time $2^{l_m-l} n\Delta t$ and $2^{l_m-l}(n+0.5)\Delta t$, respectively. With the index control operators

$$d_{l,h} \quad = \quad \mathrm{int}\left(\frac{h}{2^{l_m-l}}\right) 2^{l_m-l} \tag{1}$$

$$\hat{d}_{l,h} \quad = \quad \left(\mathrm{int}\left(\frac{h}{2^{l_m-l}}\right) + \frac{1}{2}\right) 2^{l_m-l} \tag{2}$$

where $d$ denotes the time and space discretization indices $n$, $i$, $j$ and $k$, and int() is the integer part of the argument, the updating equation for the electric-field component ${}^l e^1{}^{n_{l,n}}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,j}}$ reads

$$\begin{aligned}
{}^l e^1{}^{n_{l,n}}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}} = \\
& \Bigg( \left( \frac{\varepsilon_0 \varepsilon_r\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}}}{\Delta t 2^{l_m-l}} + \frac{\sigma_e\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}}}{2} \right) \\
& \quad \vec{a}_1\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}} \left( \vec{a}_2\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}} \times \vec{a}_3\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}} \right) \Bigg)^{-1} \\
& \Bigg( {}^l e^1{}^{n_{l,n-1}}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}} \left( \frac{\varepsilon_0 \varepsilon_r\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}}}{\Delta t 2^{l_m-l}} - \frac{\sigma_e\,{}_{\hat{i}_{l,i}\, j_{l,j}\, k_{l,k}}}{2} \right) \\
& \quad + {}^o h^2{}^{\hat{n}_{o,n-1}}_{\hat{i}_{l,i}\, j_{o,j}\, \hat{k}_{o,k-1}}\; g_{22\; \hat{i}_{l,i}\, j_{o,j}\, \hat{k}_{o,k-1}} \\
& \quad + {}^p h^3{}^{\hat{n}_{p,n-1}}_{\hat{i}_{l,i}\, \hat{j}_{p,j}\, k_{p,k}}\; g_{33\; \hat{i}_{l,i}\, \hat{j}_{p,j}\, k_{p,k}} \\
& \quad - {}^q h^2{}^{\hat{n}_{q,n-1}}_{\hat{i}_{l,i}\, j_{q,j}\, \hat{k}_{q,k}}\; g_{22\; \hat{i}_{l,i}\, j_{q,j}\, \hat{k}_{q,k}} \\
& \quad - {}^r h^3{}^{\hat{n}_{r,n-1}}_{\hat{i}_{l,i}\, \hat{j}_{r,j-1}\, k_{r,k}}\; g_{33\; \hat{i}_{l,i}\, \hat{j}_{r,j-1}\, k_{r,k}} \Bigg)
\end{aligned} \tag{3}$$

where $g_{ii} = \vec{a}_i \cdot \vec{a}_i$ is the $i$-th diagonal element of the metric tensor at the designated position, and $o$, $p$, $q$, $r$ are the grid-levels of the corresponding h-components. The grid boundary condition leads to $l-1 \leq o, p, q, r \leq l$, that means, each level $l$-component of the electric field is determined by components of the magnetic field from the same or the next lower level. The remaining updating expressions ${}^l e^2{}^{n_{l,n}}_{i_{l,i}\, \hat{j}_{l,j}\, k_{l,j}}$ and ${}^l e^3{}^{n_{l,n}}_{i_{l,i}\, j_{l,j}\, \hat{k}_{l,j}}$ of the electric field are obtained by index-permutation in (3).

The updating equation for the magnetic field component ${}^l h^1{}^{\hat{n}_{l,n}}_{i_{l,i}\, \hat{j}_{l,j}\, \hat{k}_{l,j}}$ is given by

$$\begin{aligned}
{}^l h^1{}^{\hat{n}_{l,n}}_{i_{l,i}\, \hat{j}_{l,j}\, \hat{k}_{l,k}} = \\
& \Bigg( \left( \frac{\mu_0 \mu_r\,{}_{i_{l,i}\, \hat{j}_{l,j}\, \hat{k}_{l,k}}}{\Delta t 2^{l_m-l}} + \frac{\sigma_m\,{}_{i_{l,i}\, \hat{j}_{l,j}\, \hat{k}_{l,k}}}{2} \right)
\end{aligned}$$

$$\left. \vec{a}_{1\ i_{l,i}\ \hat{j}_{l,j}\ k_{l,k}} \left( \vec{a}_{2\ i_{l,i}\ \hat{j}_{l,j}\ \hat{k}_{l,k}} \times \vec{a}_{3\ i_{l,i}\ \hat{j}_{l,j}\ k_{l,k}} \right) \right)^{-1}$$

$$\left( {}^{l}h^{1\ \hat{n}_{l,n-1}}_{\ i_{l,i}\ \hat{j}_{l,j}\ \hat{k}_{l,k}} \left( \frac{\mu_0 \mu_{r\ i_{l,i}\ \hat{j}_{l,j}\ \hat{k}_{l,k}}}{\Delta t 2^{l_m - l}} - \frac{\sigma_{m\ i_{l,i}\ \hat{j}_{l,j}\ \hat{k}_{l,k}}}{2} \right) \right.$$
$$- \sum_{s=0}^{l-o} {}^{o}e^{2\ n_{o,n-1}}_{\ i_{l,i}\ \hat{j}_{o,j-s}\ k_{l,k}}\ g_{22\ i_{l,i}\ \hat{j}_{o,j-s}\ k_{l,k}}$$
$$- \sum_{t=0}^{l-p} {}^{p}e^{3\ n_{p,n-1}}_{\ i_{l,i}\ j_{l,j+1}\ \hat{k}_{p,k-t}}\ g_{33\ i_{l,i}\ j_{l,j+1}\ \hat{k}_{p,k-t}}$$
$$+ \sum_{u=0}^{l-q} {}^{q}e^{2\ n_{q,n-1}}_{\ i_{l,i}\ \hat{j}_{q,j-u}\ k_{l,k+1}}\ g_{22\ i_{l,i}\ \hat{j}_{q,j-u}\ k_{l,k+1}}$$
$$\left. + \sum_{v=0}^{l-r} {}^{r}e^{3\ n_{r,n-1}}_{\ i_{l,i}\ j_{l,j}\ \hat{k}_{r,k-v}}\ g_{33\ i_{l,i}\ j_{l,j}\ \hat{k}_{r,k-v}} \right) \qquad (4)$$

with $o$, $p$, $q$, $r$ being the levels of the components of the electric fields where the grid boundary condition ensures $l+1 \geq o, p, q, r \geq l$ .

For a discretization containing cells of the maximal level $l_m$, a total number of $\sum_{l=1}^{l_m+1} 2^l$ different $e$ and $h$ updated computations have to be performed to proceed one step ${}^0\Delta t = \Delta t 2^{l_m}$ in time in the coarsest grid (of level 0). This update sequence is repeated after $2^{l_m}$ time steps.

In contrast to the spatial and/or temporal interpolation techniques hitherto reported, the generalized subgrid FD-TD method presented by the equations (3), (4), (1), (2), satisfies implicitly for each cell in both the main and the dual grid the divergence relation. Analogous to the standard non subgrid FD-TD formulation, the integral contribution of each vertex of the cell appears twice but with different signs so that the sum is zero. This yields for all investigated cases stable results as has been tested even at highly resonant structures, e.g. dielectric resonator filters, with up to several millions of time iterations.

The time step in each grid level $l$ is limited by the CFL condition for the nonorthogonal FD-TD method

$$ {}^{l}\Delta t = \frac{{}^{l}d_{\min}}{c_0} \qquad (5)$$

where ${}^{l}d_{\min}$ represents the smallest slope in the level $l$ [14], i.e. the smallest distance between a cell corner of the main grid to the dual grid node inside the cell.

## III. Numerical results

The 3D FD-TD subgrid formulation presented above has been tested by the analysis of an inter-digital-filter with rounded posts. Fig. 3 shows a sketch of the structure together with the geometrical data.

The 3D subgrid discretization is illustrated in Fig. 4. Several FD-TD analysis runs have shown that the S-matrix of the structure is highly sensitive to the adequate field representation in the capacitive gaps of the cylindrical posts. Therefore, a very high discretization level in these gaps (Fig. 4) has to be chosen in order to obtain convergent results.

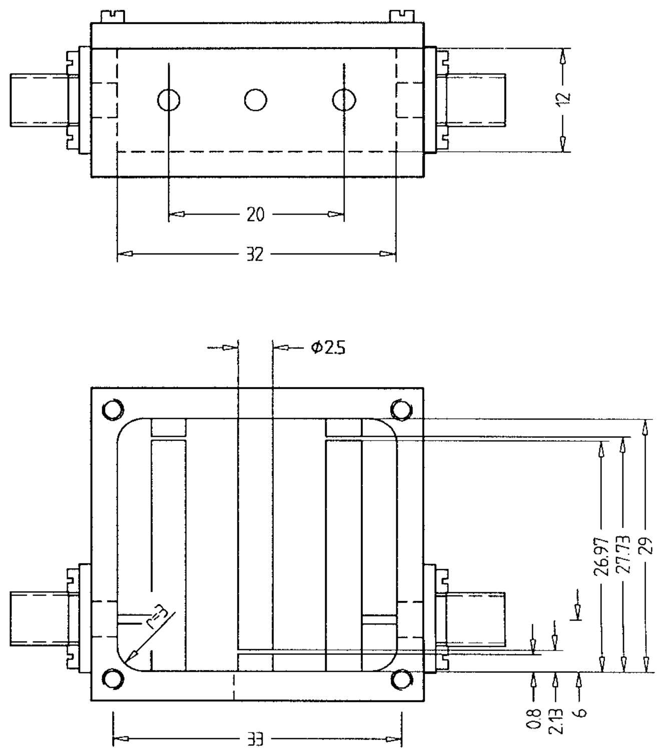
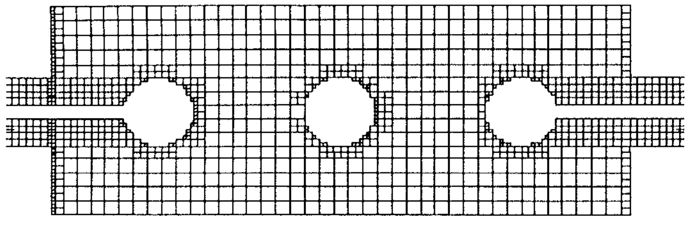



Fig. 3. Geometry of the investigated inter-digital filter. ( Dimensions in mm.)

Fig. 5 shows the S-parameters computed by the new FD-TD subgrid method compared to measurements. Good agreement may be stated. In this simulation, the matrix-pencil technique [13], [10], [9] has been successfully applied to the low-pass-filtered time-domain signals in order to reduce the total number of FD-TD time steps and to obtain frequency-domain data with arbitrary resolution.

## IV. Conclusion

A new generalized finite-difference time-domain subgrid (FD-TDS) technique has been presented. The method is based on an orthogonalization of the discretization geometry and uses the integral-form of Maxwell's curl equations. At the grid interfaces no special cases and no additional interpolations have to be considered. Analogous to the standard-FD-TD-method, the divergence conditions are satisfied in each cell of both the main and the dual grid. This yields stable results in all investigated cases.



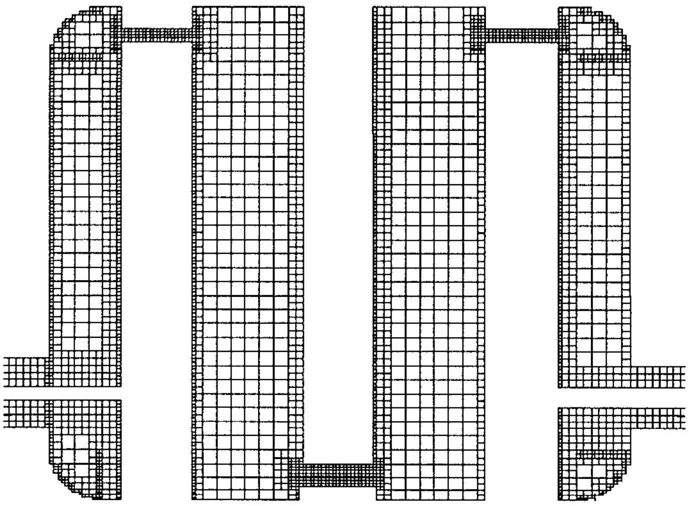


Fig. 4. Subgrid discretization of the investigated inter-digital filter

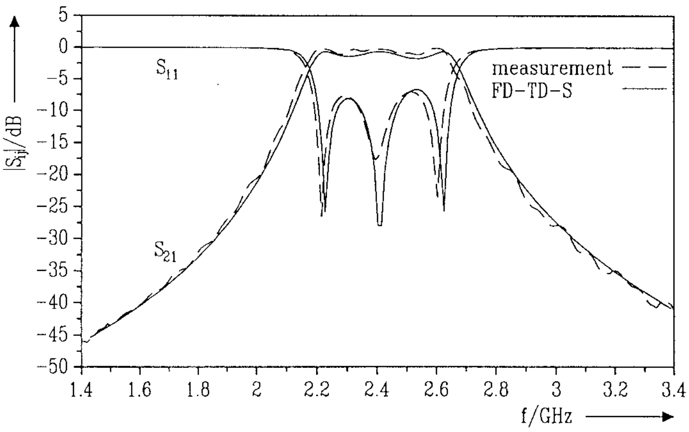



Fig. 5. S-parameters of the inter-digital filter (Fig. 4). Theory and measurements.

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